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Supplement to Linear Multistep Methods for Volterra Integral and Integro-Differential Equations

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In these appendices we present, successively,

I conditions for the existence of a unique solution of (1.1) and (1.2);

II three tables of coefficients of forward differentiation formulas, and of two common LM formulas for ODEs, viz., backward differentiation formulas and Adams-Moulton formulas;

III two lemmas which are needed in:

IV proofs of the main results of this paper, as far as they are non-trival (in the opinion of the authors).

APPENDIX I

Conditions for the existence of a unique solution $y(t) \in C(I)$ *of* (1.1) *with* $\theta = 1$

- $K(t,\tau,y)$ is continuous with respect to t and τ, for all $(t,\tau) \in S$;
- K satisfies a (uniform) Lipschitz condition with respect to y, i.e., $|K(t,\tau,y) - K(t,\tau,z)| \le L_1|y-z|$, for all $(t,\tau) \in S$, for all finite $y,z \in \mathbb{R}$;
- $g(t) \in C(I)$. □

Conditions for the existence of a unique solution $y(t) \in C(I)$ *of* (1.1) *with* $\theta = 0$

- $K(t,\tau,y) \in C^1(S\times\mathbb{R})$;
- for $t = \tau$ the derivative $\partial K/\partial y$ is bounded away from zero: $|\partial K(t,t,y)/\partial y| \ge r_0 > 0$ for all $t \in I$, $y \in \mathbb{R}$;
- $\partial K(t,\tau,y)/\partial t$ satisfies a (uniform) Lipschitz condition with respect to y on $S \times \mathbb{R}$;
- $g(t) \in C^1(I)$ with $g(t_0) = 0$. □

Conditions for the existence of a unique solution $y(t) \in C^1(I)$ *of* (1.2), *for given initial value* $y(t_0) = y_0$

The following three (uniform) Lipschitz conditions:

- $|f(t,y_1,z) - f(t,y_2,z)| \le L_1|y_1-y_2|$, for all $t \in I$, for all finite $z,y_1,y_2 \in \mathbb{R}$;

- $|f(t,y,z_1) - f(t,y,z_2')| \le L_2|z_1-z_2|$, for all $t \in I$, for all finite $y, z_1, z_2 \in \mathbb{R}$;
- $|K(t,\tau,y_1) - K(t,\tau,y_2)| \le L_3|y_1-y_2|$, for all $(t,\tau) \in S$, for all finite $y_1, y_2 \in \mathbb{R}$. □

APPENDIX II

Table 1 Coefficients of forward differentiation formulas

$$f'(t_n) \approx \frac{-1}{h} \sum_{\ell=0}^{k} \delta_\ell f(t_{n+\ell}), \quad t_{n+\ell} = t_n + \ell h$$

k	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5
1	1	-1				
2	3/2	-2	1/2			
3	11/6	-3	3/2	-1/3		
4	25/12	-4	3	-4/3	1/4	
5	137/60	-5	5	-10/3	5/4	-1/5

Table 2 Coefficients of the backward differentiation formulas

for ODEs $f'(t) = g(t)$: $\sum_{i=0}^{k} a_i f_{n-i} = b_0 g_n$

k	a_0	a_1	a_2	a_3	a_4	a_5	b_0
1	1	-1					1
2	1	-4/3	1/3				2/3
3	1	-18/11	9/11	-2/11			6/11
4	1	-48/25	36/25	-16/25	3/25		12/25
5	1	-300/137	300/137	-200/137	75/137	-12/137	60/137

Table 3 Coefficients of the Adams-Moulton formulas

for ODEs $f'(t) = g(t)$: $f_n - f_{n-1} = \sum_{i=0}^{k} b_i g_{n-i}$

k	b_0	b_1	b_2	b_3	b_4	b_5
1	1/2	1/2				
2	5/12	2/3	-1/12			
3	3/8	19/24	-5/24	1/24		
4	251/720	323/360	-11/30	53/360	-19/720	
5	95/288	1427/1440	-133/240	241/720	-137/1440	3/160

APPENDIX III

LEMMA A.1. *Let* $z_n \ge 0$ *for* $n = 0,1,\dots,N$, *and suppose that*

$$z_n \le hC_1 \sum_{i=0}^{n-1} z_i + C_2, \quad n = k,k+1,\dots,N,$$

where $k > 0$, $h > 0$ *and* $C_i > 0$ $(i=1,2)$. *Suppose, moreover, that* $z_j \le z/k$ *for* $j = 0,1,\dots,k-1$. *Then*

$$z_n \le (hC_1 z + C_2)(1+hC_1)^{n-k}, \quad n = k,k+1,\dots,N.$$

PROOF. See [7].

LEMMA A.2. *Consider the linear inhomogeneous difference equation with constant coefficients* ζ_j:

$$\zeta_0 y_{n+k} + \zeta_1 y_{n+k-1} + \dots + \zeta_k y_n = g_{n+k}, \quad n \ge 0, \tag{A.1}$$

where $\{g_n\}$ *is a given sequence, independent of the* y_n.
(i) Iff the characteristic polynomial $\zeta(z) := \Sigma_{j=0}^{k} \zeta_j z^{k-j}$ *is simple von Neumann (cf. Section 2.3) then the solution of (A.1) satisfies the inequality*

$$|y_n| \le C\{\max_{0\le j\le k-1} |y_j| + \sum_{j=k}^{n} |g_j|\}, \quad n \ge k,$$

where C *is independent of* n.
(ii) Iff $\zeta(z)$ *is Schur (cf. Section 2.3) then the solution of (A.1) satisfies the inequality*

$$|y_n| \le C\{\max_{0\le j\le k-1} |y_j| + \max_{k\le j\le n} |g_j|\}, \quad n \ge k,$$

where C *is independent of* n.

PROOF. See [7].

APPENDIX IV

PROOF OF THEOREM 2.2.1. Taylor expansion of $Y(t_{n+j}, t_{n-i})$ around (t_n, t_n) yields

$$\underset{\sim}{L}_n[Y] = \sum_{i=0}^{k} \Big\{\alpha_i \sum_{q=0}^{p} \frac{1}{q!} h^q\big(-i\frac{\partial}{\partial t} - i\frac{\partial}{\partial s}\big)^q Y(t,s)$$

$$+ \sum_{j=-k}^{k} [\beta_{ij} - \gamma_{ij} h\frac{\partial}{\partial s}] \sum_{q=0}^{p} \frac{1}{q!} h^q\big(j\frac{\partial}{\partial t} - i\frac{\partial}{\partial s}\big)^q Y(t,s)\Big\}\Big|_{(t_n,t_n)}$$

$$+ O(h^{p+1}) \text{ as } h \to 0.$$

Writing this formula in the form

$$\underset{\sim}{L}_n[Y] = \sum_{q=0}^{p} \frac{1}{q!} h^q (D_q Y)\big|_{(t_n,t_n)} + O(h^{p+1})$$

and expanding the differential operator D_q by the binomial theorem we find

$$D_q = \sum_{i=0}^{k} \Big\{\alpha_i \big(-i\frac{\partial}{\partial t} - i\frac{\partial}{\partial s}\big)^q + \sum_{j=-k}^{k} [j\beta_{ij}\frac{\partial}{\partial t} - (i\beta_{ij} + q\gamma_{ij})\frac{\partial}{\partial s}][j\frac{\partial}{\partial t} - i\frac{\partial}{\partial s}]^{q-1}\Big\}$$

$$= \sum_{\ell=0}^{q} \sum_{i=0}^{k} \Big\{(-i)^q \alpha_i - \sum_{j=-k}^{k} j^{q-\ell}(-i)^{\ell-1}[i\beta_{ij} + \ell\gamma_{ij}]\Big\}\binom{q}{\ell}\big(\frac{\partial}{\partial t}\big)^{q-\ell}\big(\frac{\partial}{\partial s}\big)^{\ell},$$

where $(-i)^{\ell-1}\ell$ is assumed to be zero for $i = \ell = 0$. Equating to zero all terms in the $\Sigma_{\ell=0}^{q}$ yields the order equations (2.2.3) and at the same time $\underset{\sim}{L}_n(Y) = O(h^{p+1})$ as required in Definition 2.2.1. □

PROOF OF THEOREM 2.2.2. Taylor expansion of $Y(t_{n+j}, t_{n-i})$ around (t_n, t_n) yields

$$Y(t_{n+j}, t_{n-i}) = \sum_{q=0}^{p} \frac{1}{q!} h^q [j\frac{\partial}{\partial t} - i\frac{\partial}{\partial s}]^q Y(t,s)\big|_{(t_n,t_n)} +$$

$$+ O(h^{p+1}) \text{ as } h \to 0.$$

In order to exploit the fact that $Y(t,t) \equiv 0$ (see definition 2.2.1), we introduce the variables $u = t + s$ and $v = t - s$ and write

$$Y(t,s) = Y\big(\frac{u+v}{2}, \frac{u-v}{2}\big) =: Z(u,v).$$

The identity $Y(t,t) \equiv 0$ implies that Z and all its derivatives with respect to u vanish for $u = 2t$ and $v = 0$. In the following we use the notation

$$Z^{(n,m)} := \frac{\partial^n \partial^m Z}{\partial u^n \partial v^m}(2t_n, 0).$$

By means of the binomial theorem we have

$$\text{(A.2)} \qquad Y(t_{n+j}, t_{n-i}) = \sum_{q=0}^{p} \frac{1}{q!} h^q [(j-i)\frac{\partial}{\partial u} + (j+i)\frac{\partial}{\partial v}]^q Z(u,v)\big|_{(2t_n,0)} + O(h^{p+1})$$

$$= \sum_{q=0}^{p} \sum_{\ell=0}^{q} \frac{1}{q!} h^q \binom{q}{\ell} (j-i)^{q-\ell} (j+i)^{\ell} Z^{(q-\ell,\ell)} + O(h^{p+1}) \text{ as } h \to 0$$

and

$$\text{(A3)}\quad hY_s(t_{n+j},t_{n-i}) = \sum_{q=0}^{p}\sum_{\ell=0}^{q}\frac{1}{q!}h^{q+1}\binom{q}{\ell}(j-i)^{q-\ell}(j+i)^{\ell}[z^{(q-\ell+1,\ell)} - z^{(q-\ell,\ell+1)}] + O(h^{p+1})$$

$$= \sum_{q=0}^{p}\sum_{\ell=0}^{q}\frac{1}{q!}h^{q}\binom{q}{\ell}(j-i)^{q-\ell-1}(j+i)^{\ell-1}[qj+qi-2\ell j]z^{(q-\ell,\ell)} + O(h^{p+1}) \text{ as } h \to 0.$$

Substitution of (A.2) and (A.3) into $\underset{\sim}{L}_n[Y]$ and using $z^{(q,0)} = 0$ yields

$$\underset{\sim}{L}_n[Y] = \sum_{q=1}^{p}\frac{1}{q!}h^q\sum_{\ell=1}^{q}\binom{q}{\ell}B_{q\ell}z^{(q-\ell,\ell)} + O(h^{p+1}) \text{ as } h \to 0$$

where $B_{q\ell}$ is defined in (2.2.4). This proves the theorem. □

PROOF OF THEOREM 2.3.1.

PROOF. Taylor expansion in a fixed point $t = t_n$ yields, respectively,

$$y(t_{n-i}) = \sum_{q=0}^{m}\frac{1}{q!}\left(-ih\frac{d}{dt}\right)^q y(t_n) + O(h^{m+1}),$$

$$Y_{n-i}(t_{n+j}) = Y(t_{n+j},t_{n-i}) - E_{n-i}(h;t_{n+j})$$

$$= \sum_{q=0}^{m}\frac{1}{q!}h^q\left(j\frac{\partial}{\partial t}-i\frac{\partial}{\partial s}\right)^q Y(t_n,t_n) + O(h^r+h^{m+1})$$

$$= \sum_{q=0}^{m}\frac{1}{q!}h^q\sum_{\ell=0}^{q} j^{q-\ell}(-i)^{\ell}\binom{q}{\ell}\frac{\partial^{q-\ell}\partial^{\ell}Y}{\partial t^{q-\ell}\partial s^{\ell}}(t_n,t_n) + O(h^r+h^{m+1})$$

$$K_{n-i}(t_{n+j}) = K(t_{n+j},t_{n-i},y(t_{n-i})) = \frac{\partial}{\partial s}Y(t_{n+j},t_{n-i})$$

$$= \sum_{q=0}^{m}\frac{1}{q!}h^{q-1}\sum_{\ell=0}^{q} j^{q-\ell}(-i)^{\ell-1}\binom{q}{\ell}\ell\frac{\partial^{q-\ell}\partial^{\ell}Y}{\partial t^{q-\ell}\partial s^{\ell}}(t_n,t_n) + O(h^m).$$

From these expansions it is immediate that the VLM formula (2.1.4) satisfies the relation

$$\text{(A.4)}\quad \sum_{i=0}^{k}\left\{\alpha_i y(t_{n-i}) + \sum_{j=-k}^{k}[\beta_{ij}Y_{n-i}(t_{n+j}) - h\gamma_{ij}K_{n-i}(t_{n+j})]\right\}$$

$$= \sum_{q=0}^{m}h^q\left\{\frac{\theta A_q}{q!}\frac{d^q y}{dt^q}(t_n) + \sum_{\ell=0}^{q}\left(C_{q\ell} - \frac{A_q}{\ell!(q-\ell)!}\right)\frac{\partial^{q-\ell}}{\partial t}\frac{\partial^{\ell}}{\partial s}Y(t_n,t_n)\right\} + O(h^r+h^{m+1})$$

where A_q and $C_{q\ell}$ are defined by (2.3.2) and (2.2.3), respectively. Under the conditions of the theorem it is easily verified that this equation leads to (2.3.3). Furthermore, (2.3.3) is obviously the m-times differentiated form of equation (1.1). □

PROOF OF THEOREM 2.3.2. Let $Y(t,s)$ be given by (1.6) where $y(t)$ is the exact solution of (1.1), then we may write for $n \geq k$

$$\underset{\sim}{L}_n(Y) \equiv \underset{\sim}{L}_n(Y) - \sum_{i=0}^{k}[\alpha_i y_{n-i} + \sum_{j=-k}^{k}(\beta_{ij}Y_{n-i}(t_{n+j}) - h\gamma_{ij}K_{n-i}(t_{n+j}))]$$

$$= \sum_{i=0}^{k}\left\{\alpha_i\varepsilon_{n-i} + \sum_{j=-k}^{k}[\beta_{ij}(Y(t_{n+j},t_{n-i}) - Y_{n-i}(t_{n+j})) - h\gamma_{ij}(K(t_{n+j},t_{n-i},y(t_{n-i})) - K_{n-i}(t_{n+j}))]\right\}.$$

Substitution of the functions $Y(t,s)$ and $Y_n(t)$ and using (2.1.3) and (2.3.6b) leads to

$$\text{(A.5)} \qquad \underset{\sim}{L}_n(Y) = \sum_{i=0}^{k}\Big\{\alpha_i\varepsilon_{n-i} + \sum_{j=-k}^{k}[\beta_{ij}\Big(h\sum_{\ell=0}^{n-i} w_{n-i,\ell}\Delta K(t_{n+j},t_\ell,y(t_\ell),y_\ell)$$

$$+ E_{n-i}(h;t_{n+j})\Big) - h\gamma_{ij}\Delta K(t_{n+j},t_{n-i},y(t_{n-i}),y_{n-i})]\Big\}.$$

Thus, we have found for the errors ε_n the relation

$$\text{(A.6)} \qquad \sum_{i=0}^{k}\alpha_i\varepsilon_{n-i} = v_n, \quad n \ge k^*, \text{ where}$$

$$v_n = \underset{\sim}{L}_n(Y) - \sum_{i=0}^{k}\sum_{j=-k}^{k}\Big[h\beta_{ij}\sum_{\ell=0}^{n} w_{n-i,\ell}\Delta K(t_{n+j},t_\ell,y(t_\ell),y_\ell)$$

$$+ \beta_{ij}E_{n-i}(h;t_{n+j}) - h\gamma_{ij}\Delta K(t_{n+j},t_{n-i},y(t_{n-i}),y_{n-i})\Big].$$

We now proceed with the two cases (a) and (b) separately.

(a) $\alpha(z) \equiv \alpha_0 z^k, \quad \alpha_0 \neq 0.$

We want to apply the discrete Gronwall inequality stated in Lemma A.1 in order to derive an upper bound for the solution of this linear difference equation, and therefore we need an upper bound for $|v_n|$. A straightforward calculation yields

$$\text{(A.7)} \qquad |v_n| \le T(h) + \sum_{i=0}^{k}\sum_{j=-k}^{k}[bwL_1h\sum_{\ell=0}^{n}|\varepsilon_\ell| + cL_1h|\varepsilon_{n-i}| + bE(h)]$$

$$\le C_0h\sum_{\ell=0}^{n}|\varepsilon_\ell| + C_1E(h) + T(h),$$

where C_0 and C_1 are constants independent of h and n (in the following all constants C_j will be independent of h and n). From (A.6) it follows that

$$|\alpha_0||\varepsilon_n| \le C_0h\sum_{\ell=0}^{n}|\varepsilon_\ell| + C_1E(h) + T(h)$$

so that for h sufficiently small

$$|\varepsilon_n| \le \frac{1}{|\alpha_0|-C_0h}[C_0h\sum_{\ell=0}^{n-1}|\varepsilon_\ell| + C_1E(h) + T(h)]$$

$$\le C_2h\sum_{\ell=0}^{n-1}|\varepsilon_\ell| + C_3[E(h) + T(h)].$$

Application of Lemma A.1 (with $z=k^*\delta(h)$) yields

$$|\varepsilon_n| \le (1+C_2h)^{n-k^*}(k^*hC_2\delta(h)+C_3[E(h)+T(h)]),$$

$$n = k^*,\ldots,N.$$

Since $nh \le T - t_0$, part (a) of the theorem is immediate.

(b) $\alpha(z)$ is simple von Neumann, $\beta(z) \equiv 0$.

Instead of directly applying Lemma A.1 to the inequality (obtained from (A.6))

$$\sum_{i=0}^{k}|\alpha_i||\varepsilon_{n-i}| \le |v_n|,$$

we first apply Lemma A.2 (i) to obtain the "sharper" inequality

$$\text{(A.8)} \qquad |\varepsilon_n| \le C_0[\delta(h) + \sum_{j=k}^{n}|v_j|], \quad n \ge k^*.$$

Unfortunately, if we use the upper bound (A.7) for $|v_j|$ and then apply Lemma A.1, we cannot prove convergence. However, by using the property $\beta(z) \equiv 0$, that is $\beta_i = \Sigma_{j=-k}^{k}\beta_{ij} = 0$, a sharper upper bound than (A.7) can be derived. To that end we write

$$\Big|\sum_{j=-k}^{k}\beta_{ij}\Delta K(t_{n+j},t_\ell,y(t_\ell),y_\ell)\Big| = \Big|\sum_{j=-k}^{k}\beta_{ij}[\Delta K(t_n,t_\ell,y(t_\ell),y_\ell)$$

$$+\ \Delta K(t_{n+j},t_\ell,y(t_\ell),y_\ell) - \Delta K(t_n,t_\ell,y(t_\ell),y_\ell)]\Big|$$

$$\le bLh\sum_{j=-k}^{k}|j\varepsilon_\ell|,$$

and, similarly,

$$\Big|\sum_{j=-k}^{k}\beta_{ij}E_{n-i}(h;t_{n+j})\Big| \le b\sum_{j=-k}^{k}\Delta E(h).$$

In this way we obtain instead of (A.7) the upper bound

$$\text{(A.9)}\qquad |v_n| \le T_n(h) + \sum_{i=0}^{k}\sum_{j=-k}^{k}[bwL|j|h^2\sum_{\ell=0}^{n}|\varepsilon_\ell| + cL_1h|\varepsilon_{n-i}| + b\Delta E(h)]$$

$$\le C_1h\sum_{i=0}^{k}[|\varepsilon_{n-i}| + h\sum_{\ell=0}^{n}|\varepsilon_\ell|] + C_2\Delta E(h) + T(h).$$

Substitution into (A.8) yields the inequality

$$|\varepsilon_n| \le C_3\Big\{\delta(h) + h\sum_{j=k}^{n}\Big[\sum_{i=0}^{k}|\varepsilon_{j-i}| + h\sum_{\ell=0}^{j}|\varepsilon_\ell| + h^{-1}\Delta E(h) + h^{-1}T(h)\Big]\Big\}.$$

It is easily verified that

$$\sum_{j=k}^{n}\sum_{i=0}^{k}|\varepsilon_{j-i}| \le (k+1)\sum_{j=0}^{n}|\varepsilon_j|.$$

Hence,

$$|\varepsilon_n| \le C_4\Big\{\delta(h) + h\Big[(1+nh)\sum_{\ell=0}^{n}|\varepsilon_\ell| + nh^{-1}\Delta E(h) + nh^{-1}T(h)\Big]\Big\}.$$

Since $nh \le T - t_0$ we find for h sufficiently small

$$|\varepsilon_n| \le C_5h\sum_{\ell=0}^{n-1}|\varepsilon_\ell| + C_6h^{-1}[h\delta(h) + \Delta E(h) + T(h)].$$

Finally, by applying Lemma A.1 we arrive at the estimate

$$|\varepsilon_n| \le (1+C_5h)^{n-k^*}\Big(k^*hC_5\delta(h)+C_6h^{-1}[h\delta(h)+\Delta E(h)+T(h)]\Big),$$

from which part (b) of the theorem follows. □

PROOF OF THEOREM 2.3.4. Following the first lines of the proof of Theorem 2.3.2 we obtain the following relation, analogous to (A.5), where $K_{rs} := K(t_r,t_s)$

$$\text{(A10)}\qquad \sum_{i=0}^{k}\sum_{j=-k}^{k}\gamma_{ij}K_{n+j,n-i}\varepsilon_{n-i} = \sum_{i=0}^{k}\sum_{j=-k}^{k}\beta_{ij}\Big[\sum_{\ell=0}^{n}w_{n-i,\ell}K_{n+j,\ell}\varepsilon_j + h^{-1}E_{n-i}(h;t_{n+j})\Big]$$

$$-\ h^{-1}\underset{\sim}{L}_n(Y),\quad n \ge k^*.$$

Now we write $K_{n+j,n-i} = K_{nn} + (K_{n+j,n-i} - K_{nn})$ and $K_{n+j,\ell} = K_{n\ell} + (K_{n+j,\ell} - K_{n\ell})$ and rewrite (A.10) to obtain

$$\text{(A.11)}\qquad \sum_{i=0}^{k}\gamma_i\varepsilon_{n-i} = v_n,\quad n \ge k^*,$$

where

$$K_{nn}v_n = h\sum_{i,j}\gamma_{ij}\Big(\frac{K_{nn}-K_{n+j,n-i}}{h}\Big)\varepsilon_{n-i} + \sum_{i,j}\beta_{ij}\sum_\ell w_{n-i,\ell}K_{n\ell}\varepsilon_\ell +$$

$$+\ h\sum_{i,j}\beta_{ij}\sum_\ell w_{n-i,\ell}\Big(\frac{K_{n+j,\ell}-K_{n\ell}}{h}\Big)\varepsilon_\ell +$$

$$+\ h^{-1}\sum_{i,j}\beta_{ij}E_{n-i}(h;t_{n+j}) - h^{-1}\underset{\sim}{L}_n(Y).$$

Since $\gamma(z)$ is Schur, we may apply Lemma A.2 (ii) to (A.11) and find

$$\text{(A.12)} \qquad |\varepsilon_n| \le C\{\delta(h) + \max_{k\le j\le n} |v_j|\}, \quad n \ge k^*.$$

where C (and all subsequent C_i) is independent of h and n. So we have to find bounds on $|v_j|$. Using the conditions of the theorem, we find

$$|v_r| \le C_1 h \sum_{i,j} |\gamma_{ij}|(j+i)|\varepsilon_{r-i}| + |\sum_{i=0}^{k} \beta_i \sum_{\ell=0}^{r} w_{r-i,\ell} K_{r\ell}\varepsilon_\ell|$$

$$+ C_2 hw \sum_{i,j} j|\beta_{ij}| \sum_{\ell=0}^{r} |\varepsilon_\ell| + h^{-1}|\sum_{i,j} \beta_{ij} E_{r-i}(h;t_{r+j})| + h^{-1}|\underset{\sim}{L}_r(Y)|,$$

$$r \ge k^*.$$

Now we use the condition $\beta(z) \equiv 0$, i.e., $\beta_i = 0$, and (2.3.6a) to obtain (cf. the derivation of (A.9) in the proof of Theorem 2.3.2)

$$|v_r| \le C_3\Big\{h \sum_{i=0}^{k} |\varepsilon_{r-i}| + h \sum_{\ell=0}^{r} |\varepsilon_\ell| + h^{-1}\Delta E(h)\Big\} + h^{-1}T(h), \quad r \ge k^*,$$

$$\le C_4\Big\{h \sum_{\ell=0}^{r} |\varepsilon_\ell| + h^{-1}\Delta E(h)\Big\} + h^{-1}T(h).$$

Substituting this into (A.12) we find, for h sufficiently small,

$$|\varepsilon_n| \le C_5\Big\{\delta(h) + h^{-1}\Delta E(h) + h^{-1}T(h) + h \sum_{\ell=0}^{n-1} |\varepsilon_\ell|\Big\}$$

and application of Lemma A.1 yields the result of the theorem. □

<u>PROOF OF THEOREM 3.3.1</u>. Proceeding as in the proof of Theorem 2.3.2 we derive the relations

$$\text{(A.13)} \qquad \underset{\sim}{L}_n^*[y] = \sum_{i=0}^{k} [\alpha_i^*\varepsilon_{n-i} - h\gamma_i^* \Delta f_{n-i}],$$

$$\underset{\sim}{L}_n[y] = \sum_{i=0}^{k}\Big\{\alpha_i\eta_{n-i} + \sum_{j=-k}^{k}\Big[\beta_{ij}\Big(h\sum_{\ell=0}^{n-i} w_{n-i,\ell}\Delta K(t_{n+j},t_\ell,y(t_\ell),y_\ell)$$

$$+ E_{n-i}(h;t_{n+j})\Big) - h\gamma_{ij}\,\Delta K(t_{n+j},t_{n-i},y(t_{n-i}),y_{n-i})\Big]\Big\}.$$

The first relation is written as (cf. (A.6))

$$\text{(A.14)} \qquad \sum_{i=0}^{k} \alpha_i^*\varepsilon_{n-i} = v_n^*,$$

where v_n^* satisfies the inequality (using (1.3') and (1.3"))

$$|v_n^*| := |\underset{\sim}{L}_n^*[y] + h \sum_{i=0}^{k} \gamma_i^*\Delta f_{n-i}|$$

$$\le T_n^*(h) + h \sum_{i=0}^{k} |\gamma_i^*|\,[L_1|\varepsilon_{n-i}| + L_2|\eta_{n-i}|].$$

Application of Lemma A.2 (i) yields (because $\alpha^*(z)$ is simple von Neumann)

$$\text{(A.15)} \qquad |\varepsilon_n| \le C_0\Big[h \sum_{j=0}^{n} [\,|\varepsilon_j| + |\eta_j|\,] + \delta(h) + \sum_{j=k}^{n} T_j^*(h)\Big]$$

where C_0 is some constant independent of n and h.

For η_n we derive from the second relation in (A.13)

$$\text{(A.16)} \qquad \sum_{i=0}^{k} \alpha_i\eta_{n-i} = v_n$$

where v_n is defined as in (A.6).

(a) In the case where $\alpha(z) = \alpha_0 z^k$ we have from (A.7):

$$|\eta_n| \le C_1[E_n(h) + h \sum_{\ell=0}^{n} |\varepsilon_\ell| + T_n(h)], \quad n \ge k^*$$

for some constant C_1. Substitution into (A.15) yields

$$|\varepsilon_n| \le C_2\Big\{h \sum_{j=k}^{n} [\,|\varepsilon_j| + h \sum_{\ell=0}^{j} |\varepsilon_\ell| + E_j(h) + T_j(h) + h^{-1}T_j^*(h)] + \delta(h) + h\delta^*(h)\Big\}$$

$$\le C_3\Big\{h \sum_{j=0}^{n} |\varepsilon_j| + E_n(h) + T_n(h) + h^{-1}T_n^*(h) + \delta(h) + h\delta^*(h)\Big\}$$

where we have used that $nh \le T - t_0$. From Lemma A.1, part (a) of the theorem easily follows.

(b) Since $\alpha(z)$ is simple von Neumann, we apply Lemma A.2 (i) to (A.16) and use (A.9) (since $\beta(z)\equiv 0$) to find

$$|\eta_n| \le C_4\Big\{\delta^*(h) + \sum_{j=k}^{n}\Big[\sum_{i=0}^{k} (h|\varepsilon_{j-i}| + h^2 \sum_{\ell=0}^{j} |\varepsilon_\ell|) + \Delta E_j(h) + T_j(h)\Big]\Big\}$$

$$\le C_5\Big\{h \sum_{j=0}^{n} |\varepsilon_j| + \delta^*(h) + h^{-1}\Delta E_n(h) + h^{-1}T_n(h)\Big\}.$$

Substitution into (A.15) and applying Lemma A.1 leads to part (b) of the theorem. □

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